Springer

KSME

Journal of Mechanical Science and Technology 24 (6) (2010) 1329~1335
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-010-0336-1

Measuring displacement signal with an accelerometer †

Sangbo Han*

School of Mechanical Engineering and Automation, Kyungnam University, Masan, 631-701, Korea

(Manuscript Received September 11, 2009; Revised March 11, 2010; Accepted March 31, 2010)

Abstract

An effective and simple way to reconstruct displacement signal from a measured acceleration signal is proposed in this paper. To reconstruct displacement signal by means of double-integrating the time domain acceleration signal, the Nyquist frequency of the digital sampling of the acceleration signal should be much higher than the highest frequency component of the signal. On the other hand, to reconstruct displacement signal by taking the inverse Fourier transform, the magnitude of the significant frequency components of the Fourier transform of the acceleration signal should be greater than the 6 dB increment line along the frequency axis. With a predetermined resolution in time and frequency domain, determined by the sampling rate to measure and record the original signal, reconstructing high-frequency signals in the time domain and reconstructing low-frequency signals in the frequency domain will produce biased errors. Furthermore, because of the DC components inevitably included in the sampling process, low-frequency components of the signals are overestimated when displacement signals are reconstructed from the Fourier transform of the acceleration signal. The proposed method utilizes curve-fitting around the significant frequency components of the Fourier transform of the acceleration signal before it is inverse-Fourier transformed. Curve-fitting around the dominant frequency components provides much better results than simply ignoring the insignificant frequency components of the signal.

Keywords: Acceleration signal; Curve-fitting; Displacement signal; Inverse Fourier transform; Nyquist frequency; Sampling rate

1. Introduction

Accelerometers are the most frequently used transducers to measure vibration responses of structures. The main objective of signal analysis involved in a structural vibration test is obtaining information on amplitudes, frequencies, and phase differences of the measured accelerations. There are many techniques available in literature to extract the parameters of a measured signal, ranging from classical Fourier transform methods [1-3] to relatively sophisticated ones [4-7]. Converting measured acceleration signals in the form of velocities and displacements is necessary in cases when direct measurements are not possible [8]. While it is quite easy to extract information on the frequency components and root mean square values of the amplitudes of velocities and displacements from measured acceleration signals, it is difficult to reconstruct the corresponding time signals of structural responses in the form of velocities and displacements.

Generally, there are two methods in converting a measured time history of acceleration signal into a displacement signal [9]. One is by directly integrating the acceleration signal in the time domain. The other is by dividing the Fourier-transformed acceleration signal by the scale factor of $-\omega^2$ and taking its inverse Fourier transform. Both methods produced a significant amount of errors depending on the sampling resolution to digitize the response signals [10, 11]. To have better resolution in the time domain, there is a need to compromise coarse resolution in the frequency domain with a given number of sampling points and visa versa. Therefore, with a fixed resolution in time and frequency domain, converting high-frequency signals in the time domain and converting low-frequency signals in the frequency domain will produce biased errors. For a pure sinusoidal signal, ignoring the insignificant components of the Fourier transformed acceleration signal provides a reasonably accurate time history of the corresponding displacement signal [4]. In this paper, an improved frequency domain method is suggested to reconstruct displacement signals from measured acceleration signals that include multi-frequency components with a certain amount of damping. The technique is based on the curve-fitting method widely used in the experimental modal analysis to extract modal parameters from frequency response functions.

2. Errors in time domain reconstruction

The first source of error introduced in the reconstruction

† This paper was recommended for publication in revised form by Associate Editor Seokhyun Kim
*Corresponding author. Tel.: +82-55-249-2623, Fax.: +82-505-999-2160
E-mail address: sbhan@kyungnam.ac.kr

process is time resolution of the digitized acceleration signal. The numerical quadrature error involved in the two-point trapezoidal rule to integrate acceleration into velocity is given as follows [12].

$$E_2 = \frac{(\Delta t)^3}{12}\ddot{a}(t) \quad 0 \prec t \prec \Delta t \tag{1}$$

For a pure sinusoidal signal, the relationship between the second derivative of the acceleration and the velocity is given as

$$\ddot{a}(t) = -(2\pi f_0)^3 v(t) \tag{2}$$

where f_0 is the frequency of the signal. The Nyquist frequency of the measurement is determined by the sampling resolution Δt with the following relationship.

$$f_{Ny} = \frac{1}{2\Delta t} \tag{3}$$

Therefore, relative error in evaluating the velocity from the acceleration of a pure sinusoidal signal is given as

$$\varepsilon_v = \frac{\pi^3}{12}\left(\frac{f_0}{f_{Ny}}\right)^3 \tag{4}$$

In addition, the relationship between the frequency of the signal to be analyzed within a certain amount of error and the Nyquist frequency of the measurement is determined as follows.

$$f_0 = \sqrt[3]{\frac{12\varepsilon_v}{\pi^3}} f_{Ny} = 0.7287\sqrt[3]{\varepsilon_v} f_{Ny} \tag{5}$$

For example, if we want to reconstruct velocity signal by directly integrating a sinusoidal acceleration signal within 5% error, the highest frequency component to be reconstructed is $0.2685 f_{Ny}$ and within 1% of error, the highest frequency component of the signal should be less than $0.1570 f_{Ny}$. The frequency component will be much lower if the integration is performed again to reconstruct the displacement signal.

The time domain reconstruction scheme was examined with numerically generated acceleration signals of the free vibration response of 1 degree-of–freedom(DOF) system. Fig. 1 and 2 represent the displacement signals reconstructed from acceleration signals whose natural frequencies are 20.3 and 400 Hz, respectively, with a damping ratio of 1% by double integration using the trapezoidal rule in the time domain. It is assumed that acceleration signals are measured with a digital signal analyzer having 2,048 sampling points. Since record time is fixed at 1 second, time and frequency domain resolution of the digitized signals are 1/2,047 second and 1 Hz, respectively, and the Nyquist frequency is 1023.5 Hz. Figs. 1 and 2 show that the higher the frequency of the signals, the worse the reconstructed displacement signals. Another aspect, as shown by the result in Fig. 2, is that the digitized theoretical displacement signal itself is not a good representative of the original pure sinusoidal signal because of the poor sampling resolution. In this example, total sampling points is 2,048, and there are only 5 (2,048/400) points available to represent one period of the signal. Therefore, no coarse sampling resolution is recommended if the aim is to reconstruct the time domain displacement signal from a signal measured with accelerometer.

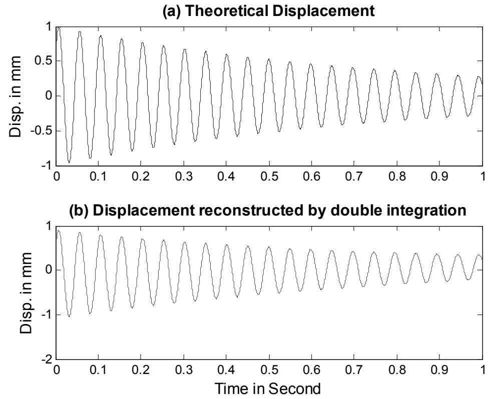


Fig. 1. Free vibration response of 1 DOF system with A natural frequency of 20.3 Hz and 1% damping.

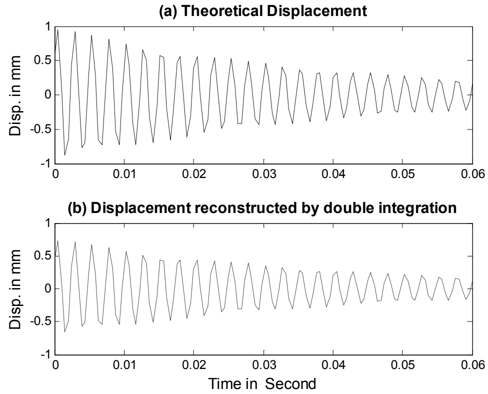


Fig. 2. Segment of free vibration response of 1 DOF system with a natural frequency of 400 Hz and 1% damping.

The second source of error is a result of unavailable information regarding the initial conditions involved in each integration scheme. The uncertain value of the initial velocity will produce a DC component during the successive integration of the conversion process (Fig. 3). One methods to eliminate the error due to the uncertain initial value of the signal is by filtering out the DC component in every integration scheme or extrapolating the acceleration signal to determine the appropriate initial velocity. The result of Fig. 3 can be improved using suitable initial velocity and displacement, which can be derived from the extrapolating scheme and the successive elimination of the DC component (Fig. 1).

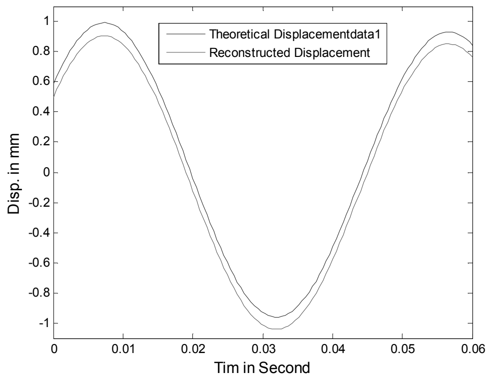


Fig. 3. Time history of the reconstructed displacement with unknown initial conditions from an acceleration signal in Fig. 1.

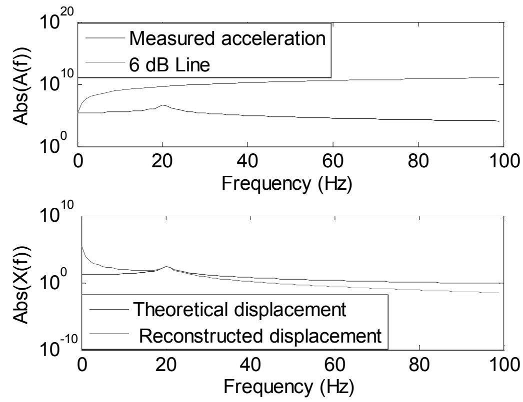


Fig. 4. Fourier transform of the free vibration response of 1 DOF system with a natural frequency of 20.3 Hz and 3% damping.

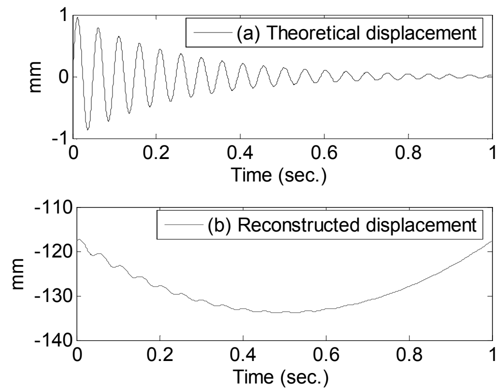


Fig. 5. Reconstructed displacement signal using the Fourier transform shown in Fig. 4(b).

3. Errors in frequency domain reconstruction

From the properties of Fourier transform of the integrals, the discrete Fourier transform of the velocity and the displacement signals is given as follows:

$$V_k = \frac{1}{j2\pi k} A_k \qquad k = 0, 1, 2, \cdots, (N-1) \tag{6}$$

$$D_k = -\frac{1}{(2\pi k)^2} A_k \qquad k = 0, 1, 2, \cdots, (N-1) \tag{7}$$

The time histories of the velocities and the accelerations are obtained by taking the inverse Fourier transform of the coefficients in Eqs. (6) and (7)

$$v_r = \sum_{k=0}^{N-1} V_k \, e^{j(2\pi kr/N)} \qquad r = 0, 1, 2, \cdots, (N-1) \tag{8}$$

$$d_r = \sum_{k=0}^{N-1} D_k \, e^{j(2\pi kr/N)} \qquad r = 0, 1, 2, \cdots, (N-1) \tag{9}$$

The error involved in the transformation of velocities and displacements comes from the scale factor of $1/j2\pi k$ and $-1/(2\pi k)^2$ in Eqs. (6) and (7). This error can be explained by the results of the signal given in Fig. 1. Theoretically, the Fourier transform of a single frequency signal is the delta function of $\delta(f - f_0)$, where f_0 is the frequency of the signal. Therefore, all the other Fourier coefficients are zero except at the corresponding frequency value. However, due to the digitization error, each Fourier coefficient has a small value, when the Fourier coefficient of the frequency component of the signal is divided by the scale factor, there is a distortion in the Fourier coefficients along the frequency axis. The amount of distortion is severer when there is DC component, leakage, or damping in the measured signal. In this case, the difference between the maximum and minimum value of the Fourier coefficients is relatively small such that the scale factor plays significant role in the conversion process.

To convert acceleration into displacement in the frequency domain, each component of the Fourier coefficients of the acceleration signal should be divided by $-(2\pi f)^2$, indicating that the value of the Fourier coefficients of the acceleration decreases inversely proportional to the square of the frequency value after being converted into displacement. When expressed in dB scale, this decrease is equivalent to 6 dB decrement. Therefore, any frequency component whose magnitude appears below the 6 dB line in the spectrum of the original acceleration signal would be much less than the insignificant noise level when converted into displacement. Thus, the significant frequency components included in the raw data signal, as well as the possibility of the distortion in the resulted converted displacement signal can be easily identified.

Suppose the measured signal has a very high frequency component, then the discrete Fourier coefficients of the acceleration will be divided by the scale factor of $-(2\pi f)^2$ with high value of f before it is converted into displacement. This will cause the significant frequency component of the signal to become less than the DC and low-frequency noise components. Fig. 4(a) represents the magnitudes of the discrete Fourier transform of the acceleration response signal corresponding to a single degree of freedom system whose natural frequency is 20.3 Hz and damping ratio is 3%. A certain level of leakage is expected because the frequency resolution of the measurement is 1 Hz. The magnitude of the 20.3 Hz component clearly appears below the 6 dB line [Fig. 4(a)], therefore, the magnitude of this frequency component becomes much less than the magnitude of the low-frequency components [Fig. 4(b)]. Taking the inverse Fourier transform of these Fourier coefficients will distort the time history of the displacement (Fig. 5).

4. Reconstruction of multi-frequency component signal

As stated above, both frequency domain and time domain methods provide a certain amount of errors depending on the frequency components of the signal. The frequency domain method works well with the acceleration signal measured without leakage, in which case the magnitude of the significant frequency component is much bigger than the non-contributing frequency components. On the other hand, the time domain method works well with the acceleration signal whose frequency is well below the Nyquist frequency. However, these conditions are seldom satisfied in real situations. In practice, structural response consists of both high-frequency and low-frequency component signals, and leakage and DC component always occur in the measurement.

When a damped signal is measured, the magnitude of the significant frequency components becomes less than that of the noise components after the Fourier coefficients are divided by the scale factor of $-\omega^2$. This results in a completely distorted time history of the displacement signal due to the abnormally exaggerated low-frequency components, as shown in Fig. 5.

To measure the structural responses with an accelerometer and to reconstruct the displacement signal after the measurement, the time domain method is preferred. However, in this case, time sampling should be fine enough to have a much higher Nyquist frequency than the highest frequency component expected in the signal to be measured.

To reconstruct a multi-frequency component signal with minimum error in the frequency domain, there must be some procedure to minimize the effects of the DC component and leakage of the measurement. The effect of the DC component and leakage on the conversion error comes from the fact that these components can become significant after the Fourier coefficients are divided by the scale factor of $-\omega^2$.

The zero-padding method involves neglecting all unwanted frequency components of the signal. Assuming that every other signal component is noise except for the significant ones appearing in the Fourier transform of the acceleration, zero padding the noise components could reduce the effect of the conversion factor. Digital band pass filter can be used on the measured and recorded signal to eliminate non-significant components of the Fourier transform of the acceleration signal. However, adjusting the band width of the filter is much more difficult than simply zero-padding the signal. A detailed explanation of the zero-padding method is included in the reference [11].

A 3 DOF system response was examined to determine the effectiveness of zero-padding the signal in the frequency domain method. The measured acceleration response was assumed to have natural frequencies of 80.3, 400.5 and 600.7 Hz and 1, 0.3 and 0.1% of modal damping for the corresponding modes, respectively. The Fourier transform of this acceleration signal is scaled with $-\omega^2$, and all other coefficients are set to zero except for the coefficients in the vicinity of the natural frequencies. The resulted Fourier transform is compared with the theoretical Fourier transform of the displacement signal in Fig. 6. Reconstructed time history of the displacement from this zero-padded Fourier transform is given in Fig. 7. As expected, the overall feature of the signal can be reconstructed, but not the exact time history, especially at the starting part of the signal. This is because the zero-padded Fourier transform cannot provide the phase information of the original signal.

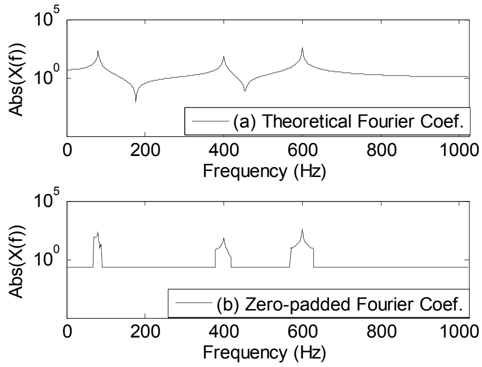


Fig. 6. Magnitude of the Fourier transform of theoretical displacement and reconstructed displacement with zero-padding method for a signal with three frequency components.

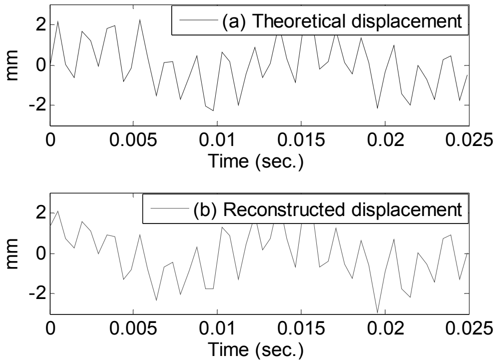


Fig. 7. Time histories of the theoretical and the reconstructed displacement using the zero-padded Fourier transform given in Fig. 6.

5. Curve-fitting the response signal

To improve the incompleteness of the reconstruction process in the frequency domain, a better procedure is proposed. Except for some special cases, most measured structural accelerations are either steady-state responses or decaying-transient responses. Steady-state responses consist of single-frequency sinusoidal signals, in which the zero-padding method can provide satisfactory results.

The decaying-transient responses of the structure, such as those of the impact hammer test can be considered as the linear combination of 1 DOF system responses. From the example of single-frequency sinusoidal signal, the reconstruction error in the frequency domain methods comes from the fact that low-frequency noise components become bigger than the actual frequency components of the signal after the Fourier transform of the acceleration signal is divided by the scale factor of $-\omega^2$. Since the acceleration response of the structure consists of the contributing modal components of the structure, any other frequency components appearing in the scaled Fourier transform can be considered as irrelevant noise components. Therefore, by extracting only the significant frequency components of the scaled Fourier transform, a reasonably accurate time history of the displacement signal can be reconstructed.

The method of extracting the significant frequency components of the signal is the same as the curve-fitting method used in the experimental modal analysis. In the experimental modal analysis, the curve-fitting method is utilized to find coefficients in a theoretical expression for the frequency response function which most closely matches the measured data. This task is most readily tackled by the series form for the frequency response function. There are a number of curve-fitting methods applicable in both frequency and time domains, and detailed procedures are given in [13]. However, the basic assumption for the various curve-fitting methods is that in the vicinity of a resonance, total response is dominated by the contribution of the mode with the closest natural frequency. In this study, the concept stating that response signal is a linear combination of sinusoidal signals whose frequency components significantly appear in the Fourier transform is utilized. By assuming that the signal is composed of individual 1 DOF system response, each single-frequency component appearing on the Fourier transformed signal can be reconstructed. In addition, the linear combination of the single-frequency components in the frequency domain is considered as the actual Fourier transform of the displacement signal to be reconstructed.

For 1DOF system, the Fourier transform of the response is given as follows.

$$X(j\omega) = \frac{\delta_{st}}{1 - r^2 + j2\zeta r} \tag{10}$$

where $X(j\omega)$ is the Fourier transform of the signal; δ_{st} is the static deflection; r is the frequency ratio; and ζ is the damping ratio of the signal. The curve-fitting procedure is applied as follows.

First, individual resonance peaks are detected on the scaled Fourier transform of the acceleration signal, and the frequency of the maximum response is taken as the significant frequency component of the signal.

Second, using non-linear curve-fitting algorithm, the δ_{st} and ζ in Eq. (10) most closely fitting the scaled Fourier transform of the signal along several frequency lines (in this study, there are nine spectral lines) around the frequency of the maximum response are determined.

Third, the theoretical expression of the Fourier transform of the response corresponding to the one in Eq. (10) is replaced with the scaled Fourier transform in order to eliminate the distorted region introduced by the scaling with $-\omega^2$. Afterward, the final Fourier transform is inverse Fourier transformed to reconstruct the time domain signal.

To evaluate the effectiveness of this method, the single DOF system response shown in Figs. 4 and 5 is tested again. As stated earlier, since the magnitude of the relevant frequency component of the acceleration response appears below the 6 dB line, simply scaling this Fourier transformed signal gives a totally incorrect time history of the displacement (Fig. 5).

The Fourier transform corresponding to the displacement signal, obtained by dividing the Fourier transform of the acceleration signal by the scale factor of $-\omega^2$ is curve-fitted taking nine spectral lines around the peak. Fig. 8 compares the curve-fitted scaled Fourier transform of the acceleration signal and the theoretical one. The contribution of the discrepancy between the curve-fitted and the theoretical Fourier transform for the frequency components far away from the peak are insignificant because of their small values. The curve-fitted Fourier transform zoomed around the peak is given in Fig. 9, and the corresponding Nyquist plot is given in Fig. 10, which demonstrates the effectiveness of this method. The results of these figures show that recovering the significant frequency components around the peak is the key factor in the reconstruction process of the displacement signal. Therefore, the effectiveness of this method depends on the accuracy of the curve-fit, which is exactly the same as the extraction of modal parameters in the experimental modal analysis.

The result of the reconstructed displacement time signal is compared with the theoretical one in Fig. 11.

In using the curve-fit method, the curve-fit algorithm may estimate a negative damping value because of the unrealistically high values of low-frequency components of the scaled Fourier transform corresponding to the displacement signal (Fig. 4). In such a case, the reconstructed time signal of the displacement appears to be diverging instead of decaying, as shown in Fig. 12. This kind of error can be easily determined by examining the Nyquist plot of the scaled Fourier transform the displacement signal, which always traces along the half plane of the negative imaginary axis (Fig. 13).

The curve-fitting method is compared with the zero-padding method by reconstructing the same 3 DOF system response given in Fig. 6. The curve-fitted Fourier transform of the displacement is given in Fig. 14. The curve-fitting method accurately provides both the overall shape and the detailed traces of the time signal as shown in Figs. 15 and 16, respectively.

Comparing both the reconstructed displacement signals using the zero-padding and curve-fitting methods in Figs. 7 and

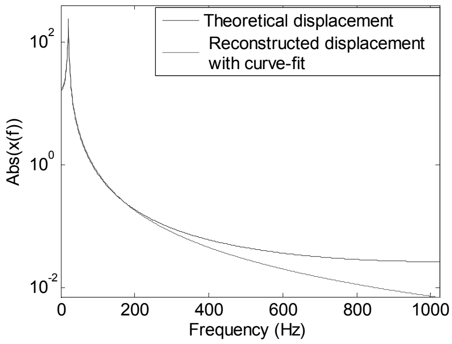


Fig. 8. Theoretical and curve-fitted Fourier transform of 1 DOF response signal.

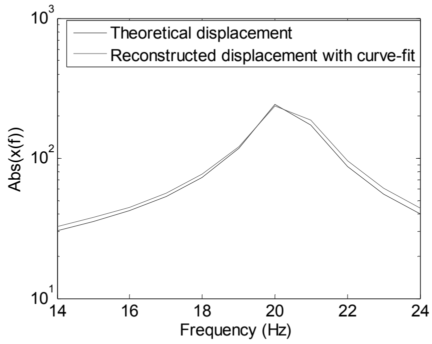


Fig. 9. Theoretical and curve-fitted Fourier transform of 1 DOF response signal zoomed around the natural frequency.

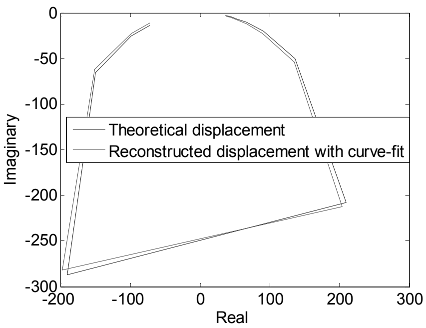


Fig. 10. Nyquist plot of the theoretical and curve-fitted Fourier transform of 1 DOF response signal.

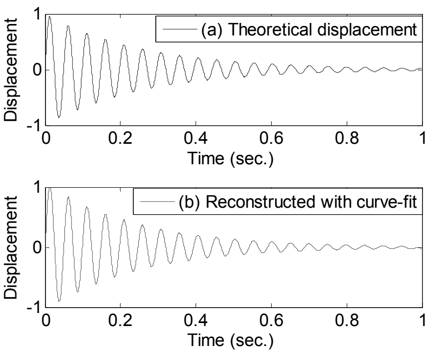


Fig. 11. Time histories of the theoretical and reconstructed displacement using the curve-fit method.

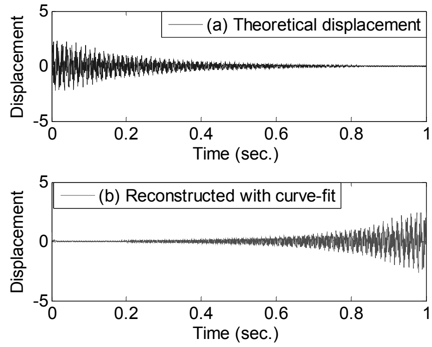


Fig. 12. Time histories of theoretical and reconstructed displacement from inaccurate curve-fit results.

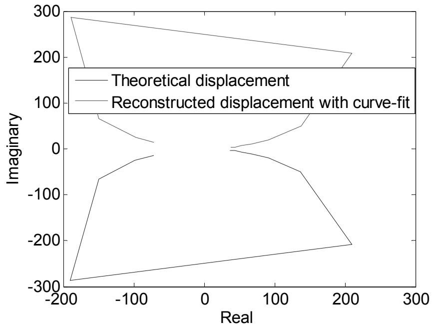


Fig. 13. Nyquist plot to be used to check the accuracy of the curve-fit method.

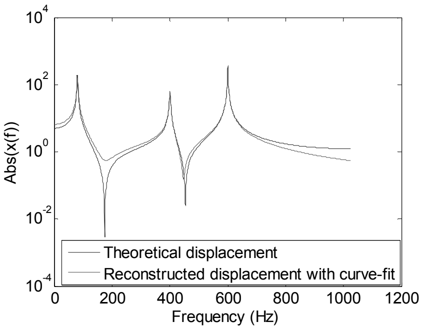


Fig. 14. Curve-fitted Fourier transform of the 3 DOF response signal to be reconstructed.

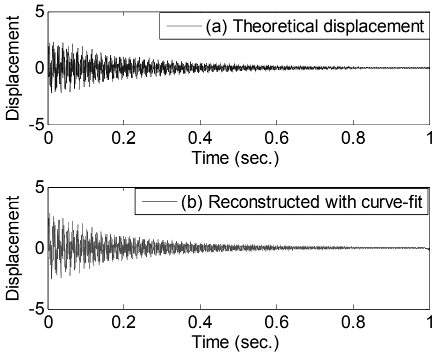


Fig. 15. Time histories of the theoretical and reconstructed displacement of 3DOF response signal using the curve-fit method in Fig. 14.

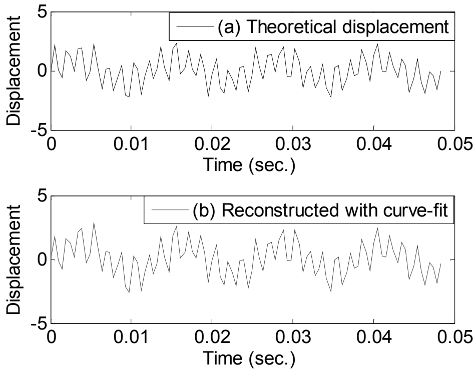


Fig. 16. Detailed traces of the time histories shown in Fig. 15.

16, the latter provides much better detailed information of the original signal. This is because the curve-fitted Fourier transform can provide the phase information of the original signal, while the zero-padded Fourier transform simply provides information of the magnitude.

6. Conclusions

There are two general ways to reconstruct the time signal of the displacement from measured acceleration signal. When the time signal of the acceleration is available and the Nyquist frequency of digitization is much higher than the highest frequency component of the signal, the direct double integration of the acceleration in the time domain provides reasonably accurate displacement time signal. When using the time domain method, the success of the reconstruction process depends on the appropriate initial conditions during the double integration process. If the frequency components of the measured acceleration signal are relatively high compared to the Nyquist frequency, the frequency domain method should be used. The effect of double integration in the time domain is achieved by scaling the discrete Fourier transform of the measured acceleration signal in the frequency domain. Curve-fitting around the peak values of the scale discrete Fourier transform provides a reasonably accurate shape of the Fourier transform of the original displacement signal. The curve-fitting method in the frequency domain is much better than the previously suggested zero-padding method, provided that the damping properties of the signal are accurately estimated.

Acknowledgment

This work is supported by the Kyungnam University Research Fund, 2009.

References

[1] J. S. Bendat and A. G. Piersol, *Random data*, 2nd Edition, John Wiley & Sons Inc., New York (1991).

[2] D. E. Newland, *An Introduction to Random Vibrations, Spectral & Wavelet Analysis*, 3rd.ed. John Wiley & Sons Inc., New York (1993).

[3] A. Papoulis, Probability, Random Variables and Stochastic *Processes*, 3rd Edition, McGraw-Hill International Edition (1991).

[4] Z. Chen and C. K. Mechefske, Machine Signature Identification by Analysis of Impulse Vibration Signals, *Journal of Sound and Vibrations*, 244 (1) (2001) 155-167.

[5] S. K. Lee, Application of the L-Wigner Distribution to the diagnosis of Local Defects of Gear Tooth, *KSME International Journal*, 13 (2) (1999) 144-157.

[6] G. Y. Luo, D. Osypiw and M. Irle, Real-Time Condition Monitoring by Significant and Natural Frequencies Analysis of Vibration Signal with Wavelet Filter and Autocorrelation Enhancement, *Journal of Sound and Vibration*, 236 (3) (2000) 413-430.

[7] S. K. Tang, On the Time-Frequency Analysis of Signals That Decay Exponentially with Time, *Journal of Sound and Vibration*, 242 (2) (2000) 241-258.

[8] B. F. Feeny and R. Kappagantu, On the Physical Interpretation of proper Orthogonal Modes in Vibrations, *Journal of Sound and Vibration*, 211 (4) (1998) 607-616.

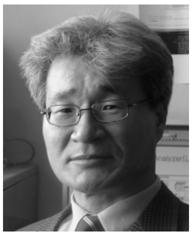
[9] K. G. McConnell, *Vibration Testing, Theory and Practice*, John Wiley & Sons Inc, New York (1995).

[10] S. Han, Analysis of Errors in the Conversion of Acceleration into displacement, Proceedings of 19th IMAC, CD ROM (2001).

[11] S. Han, Retrieving the Time History of Displacement from Measured Acceleration Signal, Journal of Mechanical Science and Technology, 17 (2) (2003) 197-206.

[12] R. L. Burden, J. D. Faires and A. C. Reynolds, *Numerical Analysis*, Prindle, Weber & Schmidt, (1979).

[13] D. J. Ewins, *Modal Testing, theory, practice and application*, 2nd ed., Research Studies Press Ltd., Baldock, England (2000).



Sangbo Han received his B.S. and M.S. degree in Naval Architecture from Seoul National University in 1981 and 1984. He then received his Ph.D. degrees from Iowa State University in 1988. He is currently a Professor at the School of Mechanical Engineering and Automation at Kyungnam University in Masan, Korea. His research interests are in the area of noise and vibration control and design of experimental system.